Proof of Lemma 6.3 in "The crossing number of $K_{4,n}$ on the torus and the Klein bottle"

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The Lemma 6.3 in "The crossing number of $K_{4,n}$ on the torus and the Klein bottle" is false, which says that

Lemma 6.3. *For any drawing D of $K_{4,n}$ on the Klein bottle, let A be the matrix defined by $A_{ij} = \widetilde{cr_D}(a_i, a_j)$. Then it is impossible for the following to hold for some distinct i_j, $1 \leq j \leq 5$:*

$$\begin{pmatrix} A_{i_1i_1} & A_{i_1i_2} & A_{i_1i_3} & A_{i_1i_4} & A_{i_1i_5} \\ A_{i_2i_1} & A_{i_2i_2} & A_{i_2i_3} & A_{i_2i_4} & A_{i_2i_5} \\ A_{i_3i_1} & A_{i_3i_2} & A_{i_3i_3} & A_{i_3i_4} & A_{i_3i_5} \\ A_{i_4i_1} & A_{i_4i_2} & A_{i_4i_3} & A_{i_4i_4} & A_{i_4i_5} \\ A_{i_5i_1} & A_{i_5i_2} & A_{i_5i_3} & A_{i_5i_4} & A_{i_5i_5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{1}$$

For the counterexample, one can refer to the following drawing of $K_{4,5}$ on the Klein bottle with $a_j = a_{i_j}$, where ▲ represents the vertices a_3, a_4 and a_5.

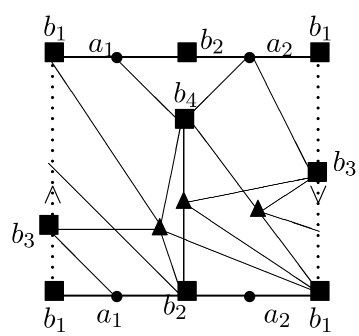


Figure 1.

APPENDIX. The possible drawings of the $K_{2,4}$ containing a_1 and a_2 in torus which has no crossings.

To obtain these drawings, we combine all the possibilities of different ways of drawing edges a_1b_3, a_2b_3, a_1b_4 and a_2b_4, as in Figure 6(i) to 6(viii) in [1]. Figure 2(s)(t) was obtained by drawing the pair a_1b_3, a_2b_3 as in Figure 6(s) in [1] and the pair a_1b_4, a_2b_4 as in Figure 6(t) in [1] where s, t equals i, ii,...., viii. For some s, t, Figure 2(s)(t) is omitted since it has crossings. Moreover, since Figure 2(s)(t) and Figure 2(t)(s) are identical, only Figure 2(s)(t), where s≤t, is shown.

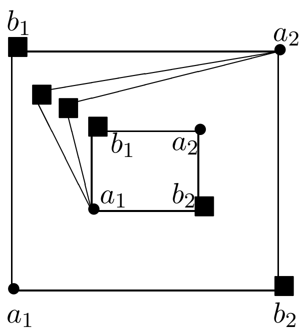


Figure 2(i)(i)

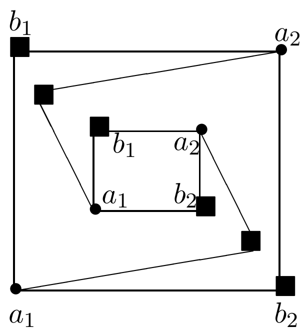


Figure 2(i)(ii)

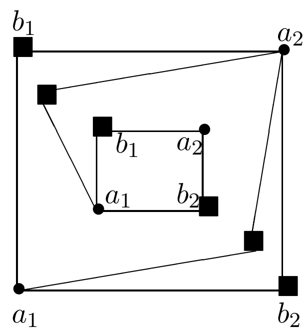


Figure 2(i)(iv)

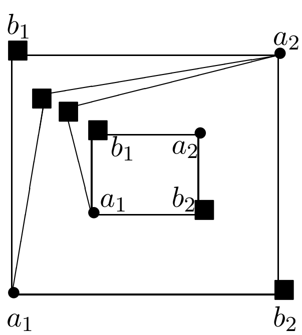


Figure 2(i)(v)

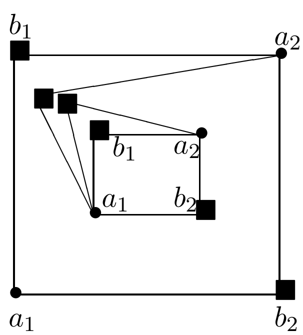


Figure 2(i)(vi)

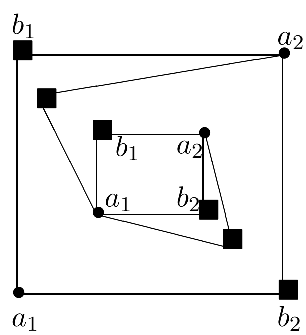


Figure 2(i)(vii)

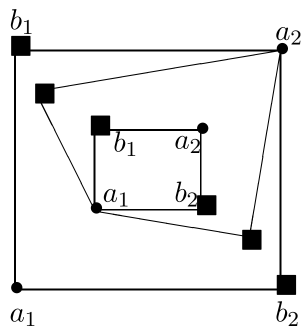
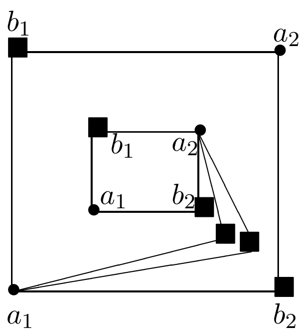


Figure 2(i)(viii)



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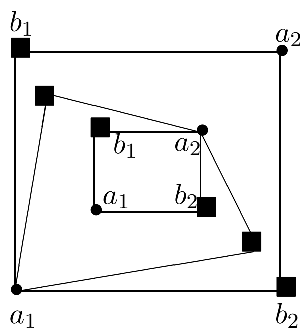

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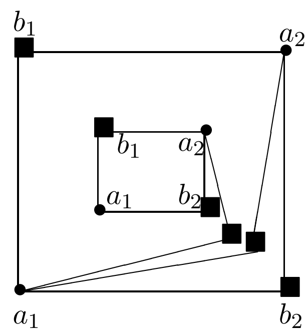
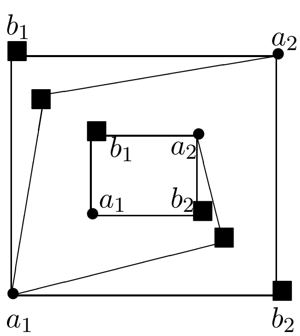


Figure 2(ii)(iv)



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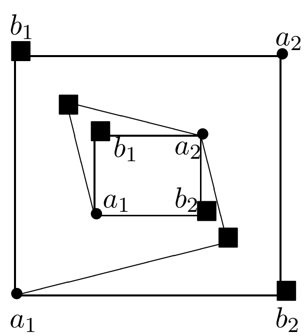

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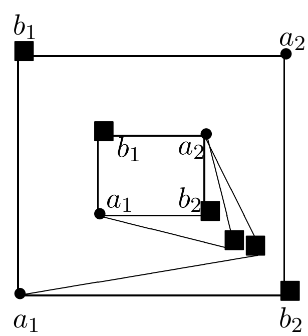
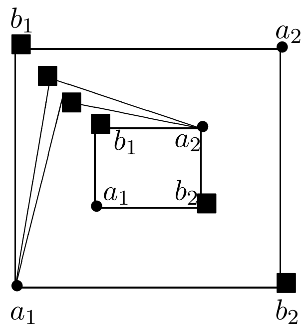


Figure 2(ii)(vii)



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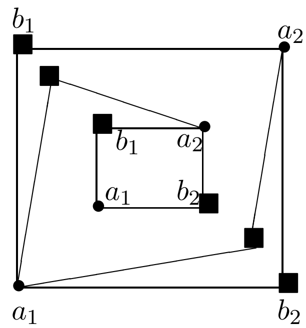

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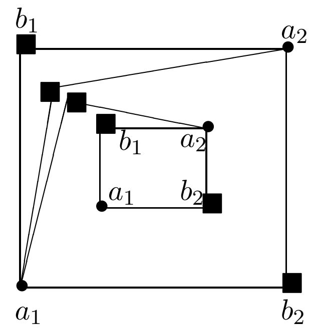
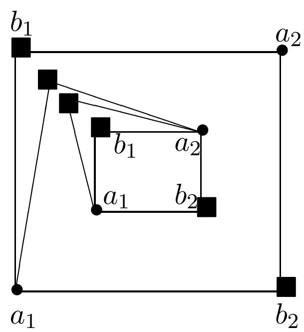


Figure 2(iii)(v)



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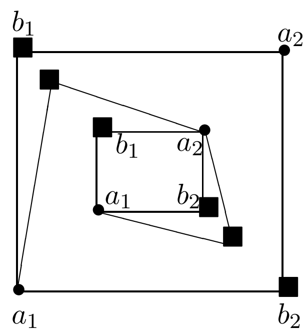

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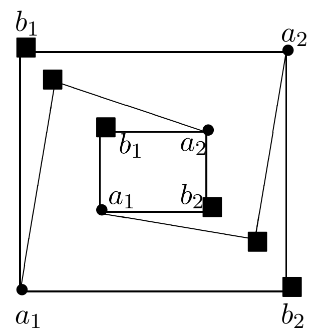

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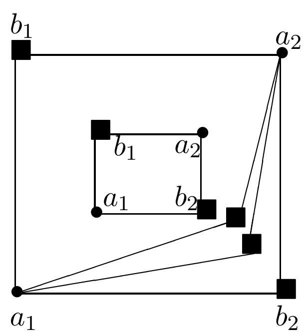

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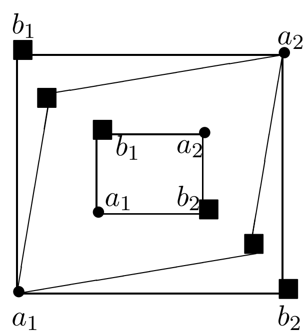

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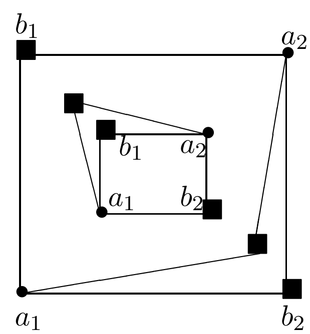


Figure 2(iv)(vi)

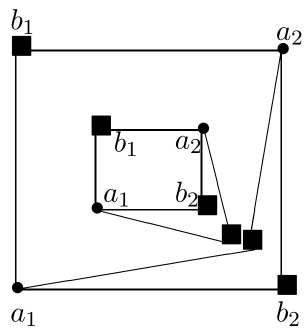
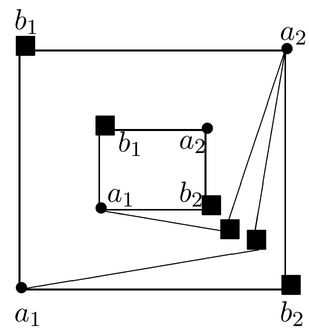


Figure 2(iv)(vii)



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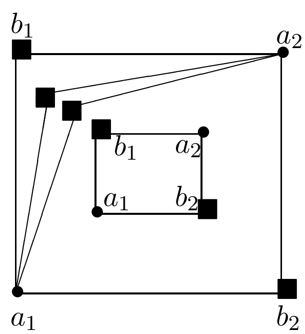


Figure 2(v)(v)

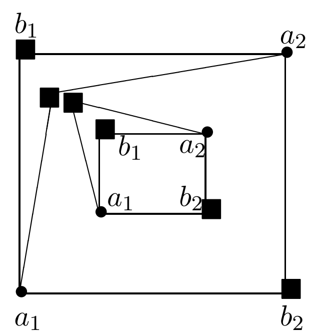
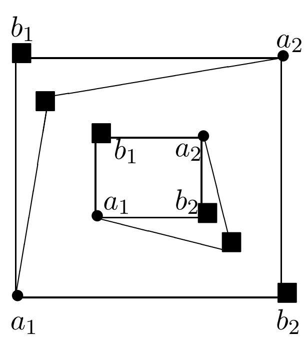


Figure 2(v)(vi)



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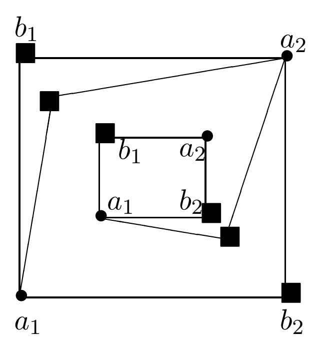

Figure 2(v)(viii)

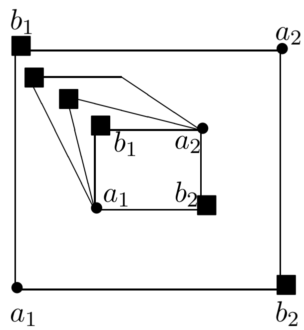

Figure 2(vi)(vi)

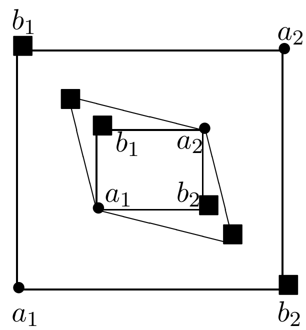

Figure 2(vi)(vii)

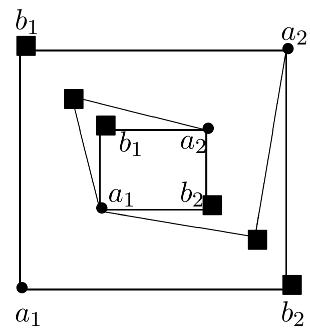

Figure 2(vi)(viii)

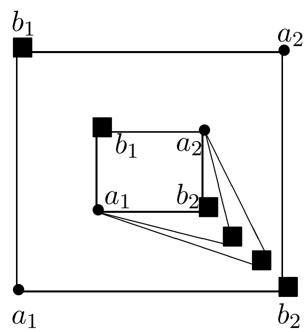

Figure 2(vii)(vii)

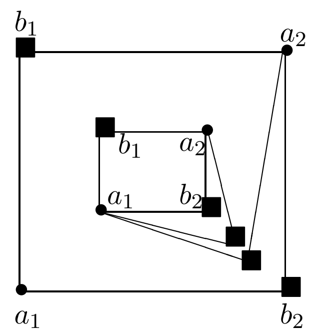

Figure 2(vii)(viii)

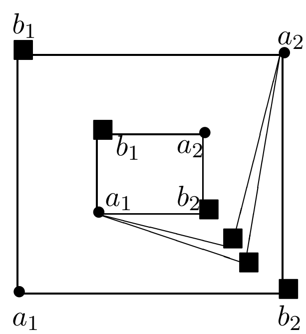

Figure 2(viii)(viii)

The following tables contain the following information: in each table, the Figure in the first row is isomorphic to the Figure in the second row which appeared in [1]. If "Not included" appears in the second row, it means that the corresponding Figure in the first row does not have a region which contains all b_i, therefore it is not included. For example, Figure 2(i)(i) is isomorphic to Figure 7(ii) in [1]; Figure 2(i)(ii) is isomorphic to Figure 7(i)

in [1]; Figure 2(i)(vi) is isomorphic to a drawing symmetric to Figure 7(i) in [1]; Figure 2(iv)(iv) is not included since it does not have a region which contains all b_i. These tables show that Figures 7(i) to 7(vii) in [1] exhaust all the possible drawings.

2(i)(i)	2(i)(ii)	2(i)(iv)	2(i)(v)	2(i)(vi)	2(i)(vii)	2(i)(viii)
7(ii)	7(i)	7(iii)	sym 7(iii)	sym 7(vi)	7(vi)	7(viii)

2(ii)(ii)	2(ii)(iii)	2(ii)(iv)	2(ii)(v)	2(ii)(vi)	2(ii)(vii)
7(ii)	7(viii)	sym 7(iii)	7(iii)	7(vi)	sym 7(vi)

2(iii)(iii)	2(iii)(iv)	2(iii)(v)	2(iii)(vi)	2(iii)(vii)	2(iii)(viii)
7(ii)	sym 7(iii)	7(iii)	7(vi)	sym 7(vi)	sym 7(i)

2(iv)(iv)	2(iv)(v)	2(iv)(vi)	2(iv)(vii)	2(iv)(viii)
Not included	7(v)	7(iv)	Not included	7(iii)

2(v)(v)	2(v)(vi)	2(v)(vii)	2(v)(viii)
Not included	Not included	7(iv)	sym 7(iii)

2(vi)(vi)	2(vi)(vii)	2(vi)(viii)
Not included	7(vii)	sym 7(vi)

2(vii)(vii)	2(vii)(viii)
Not included	7(vi)

2(viii)(viii)
7(ii)

References

[1] P. T. Ho, The crossing number of $K_{4,n}$ on the torus and the Klein bottle, *Discrete Math.*, revised.